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Gravitational peculiarities of a scalar field

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Abstract. The zero-adjoint of a time-static Ricci-flat solution to Einstein's field equations is investigated. It represents a space–time curved solely by a massless scalar field. Cylindrical symmetry is assumed, to permit both planar and non-planar geodetic motions. Unusual, velocity-dependent gravitational features are encountered from these geodesics.

1. Introduction

The study of exact solutions of gravity coupled to other fields is important to understand clearly the physical and mathematical structure of space–time (Duncan 1977). For many reasons, the coupling of scalar fields to gravitation has been an object of special attention in recent years (Bronnikov 1978, Kodama *et al* 1978, Buchdahl 1978, Chung *et al* 1977, Bekenstein 1974, 1975). In most cases, systems have been studied in which the scalar field coexists with other constituents, such as diffused matter or electromagnetic fields (Banerjee and Dutta Choudhury 1977, Teixeira *et al* 1974, 1975, 1976). In such complex systems, however, the non-linearity of the field equations generally makes it difficult to see separately the gravitational peculiarities of each constituent.

In this paper we study the gravitation associated with a massless, real scalar field, in the absence of any material source or other field. In contrast to the electromagnetic fields, the scalar field under static conditions can be described in terms of cosmic time. We consider a system with cylindrical symmetry, to permit both planar and non-planar geodesics. From the investigation of these geodesics, an interesting, velocity-dependent acceleration field is found, acting differently upon each component of the velocity vector.

2. Gravitational and scalar potentials

We are concerned with the line element

$$ds^2 = dt^2 - [(dr^2 + dz^2)r^{2b} + r^2\, d\phi^2], \qquad b = \text{constant} \geqslant 0. \tag{1}$$

It satisfies the Einstein scalar field equations

$$R_{\mu\nu} = -2\partial_\mu S \partial_\nu S, \qquad S = \pm\sqrt{b}\ln r, \tag{2}$$

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where the dimensionless constant $\pm\sqrt{b}$ represents the strength of the long-range, attractive scalar field S. The important feature of equation (1) is that it represents the gravitation from the pure scalar field, as is explained in some detail at the end of § 4.

The line element (1) can be obtained, without solving the field equations, in a variety of ways starting from the static, Ricci-flat solution with cylindrical symmetry (Weyl 1917):

$$\mathrm{d}s^2_{\mathrm{Weyl}} = (r/a)^{4\lambda}\,\mathrm{d}t^2 - (r/a)^{-4\lambda}[(\mathrm{d}r^2 + \mathrm{d}z^2)(r/a)^{8\lambda^2} + r^2\,\mathrm{d}\phi^2]. \tag{3}$$

We are using $c = G = 1$; the constant a has the dimension of length and is set equal to one, for simplicity. Following Buchdahl (1978), we should simply write the zero-adjoint of (3) and set $\lambda^2 = b/4$ to obtain equation (1). Alternatively, we can use the prescriptions of Teixeira *et al* (1976) and set the constant c^2 equal to one in their attractive scalar field. Also, we could follow the method of Janis *et al* (1969) and let their constant $A^2 \to \infty$.

3. Geodesics

To investigate the gravitational features of equation (1), we consider the geodetic differential equations

$$\ddot{t} = 0, \qquad \ddot{z} = -2b\dot{z}\dot{r}/r, \qquad \ddot{\phi} = -2\dot{\phi}\dot{r}/r, \tag{4}$$

$$r\ddot{r} = -b\dot{r}^2 + b\dot{z}^2 + r^{2(1-b)}\dot{\phi}^2, \tag{5}$$

where a dot means $\mathrm{d}/\mathrm{d}s$. With the restriction $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$, valid for time-like geodesics, we find the first integrals

$$s' = (1 - v^2)^{1/2}, \tag{6}$$

$$z' = vh/r^{2b}, \qquad \phi' = vl/r^2, \tag{7}$$

$$r' = \pm vr^{-b}(1 - h^2r^{-2b} - l^2r^{-2})^{1/2}, \tag{8}$$

where a prime means $\mathrm{d}/\mathrm{d}t$, and where the three parameters $0 \leqslant v < 1$, h and l are constants of integration.

A trival solution of equations (6)–(8) is obtained when $v = 0$, and corresponds to a particle at rest in the presence of the anisotropic fields. This interesting result is discussed in § 4. Other trivial solutions are obtained when $b = 0$, and correspond to the rectilinear, uniform motions in the flat space–time.

The non-trivial solutions of equations (6)–(8) correspond to three types of motion:

3.1. Motion on planes normal to the z axis

Setting $h = 0$ in equations (7) and (8), we obtain

$$\mathrm{d}r/\mathrm{d}\phi = \pm[(r/l)^2 - 1]^{1/2}r^{1-b}. \tag{9}$$

$|l|$ is then the minimal radial location of the particle in its motion. Table 1 presents exact solutions of equation (9), obtained for some values of the parameter b. In figure 1 are drawn some solutions corresponding to $l = 1{\cdot}5$; as in all cases where $l^2 \geqslant 1$, these solutions represent spiral motions around the z axis. In the cases where $l^2 < 1$, however, a different behaviour of the test particle is found near the z axis; in figure 2,

Table 1. Orbits $\phi(r)$ in planes z = constant, for several values of b. The functions E and F are elliptic integrals (Dwight 1961).

b	$\lvert l\rvert^{-b}\phi(r)$
0	$\sec^{-1}R \qquad (R \equiv r/\lvert l\rvert)$
$\frac{1}{2}$	$\sqrt{2}F\,(\sec^{-1}\sqrt{R},\,1/\sqrt{2}) \qquad [\equiv G(r)]$
1	$\cosh^{-1}R$
$\frac{3}{2}$	$2(R-R^{-1})^{1/2}+G(R)-2\sqrt{2}\,E(\sec^{-1}\sqrt{R},\,1/\sqrt{2})$
2	$(R^2-1)^{1/2}$
$\frac{5}{2}$	$\frac{1}{3}\{2[R(R^2-1)]^{1/2}+G(R)\}$
3	$\frac{1}{2}[R(R^2-1)^{1/2}+\cosh^{-1}R]$

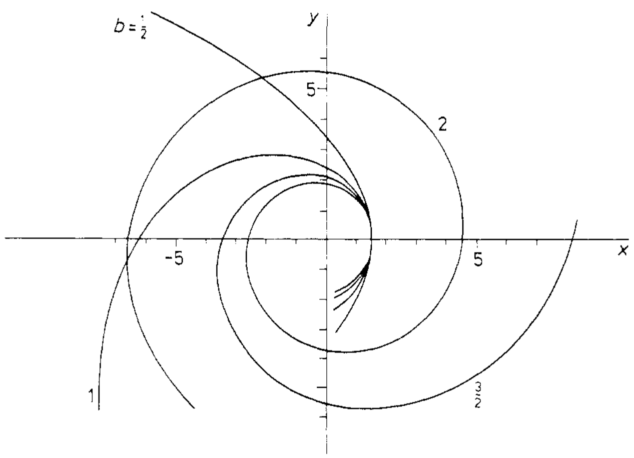

Figure 1. Orbits in planes z = constant, for $l = 1{\cdot}5$ and several values of b. Particles are attracted to the axis of symmetry, but nevertheless these always escape to the radial infinity.

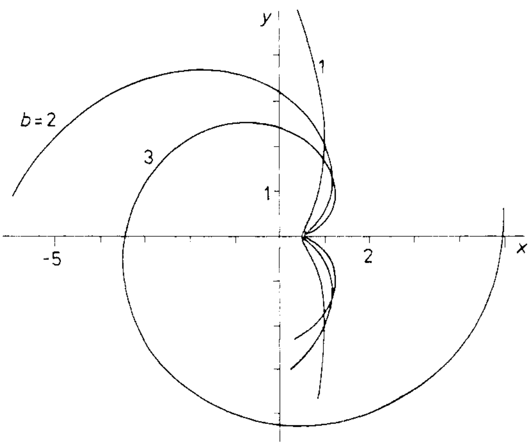

Figure 2. Orbits in planes z = constant, for $l = 0{\cdot}5$ and several values of b. Particles are repelled from the z axis for short radial distances, and attracted for larger values of r.

corresponding to motions with $l = 0{\cdot}5$, we remark that all trajectories bend outwards for small values of r.

It can be shown that the shape of orbits given by equation (9) can also be obtained from a non-relativistic, static, cylindrically symmetric potential

$$V(r) = -\tfrac{1}{2}v^2[(1-r^{-2b})(l/r)^2 + r^{-2b}]. \tag{10}$$

However, the relativistic velocity of motion in the orbit differs from its non-relativistic analogue.

3.2. *Motion on planes containing the z axis*

Setting $l = 0$ in equations (7) and (8) we obtain

$$\mathrm{d}r/\mathrm{d}z = \pm[(r/m)^{2b} - 1]^{1/2}, \tag{11}$$

where m, given by $m^b = |h|$, is the minimal value of the radial coordinate along the motion. Table 2 presents solutions of equation (11) for several values of b, while figure 3 shows the corresponding orbits. We find that the trajectory of the particle always bends outwards, which indicates repulsion from the axis of symmetry.

Table 2. Orbits $z(r)$ in planes containing the z axis, for several values of b. The functions $F(\phi, k)$ are elliptic integrals (Dwight 1961).

b	$m^{-1}z(r)$
$\frac{1}{2}$	$2(R-1)^{1/2} \qquad (R \equiv r/m)$
1	$\cosh^{-1} R$
$\frac{3}{2}$	$3^{-1/4}F\left(\cos^{-1}\frac{\sqrt{3}+1-R}{\sqrt{3}-1+R}, \sin \pi/12\right)$
2	$2^{-1/2}F(\sec^{-1} R, 1/\sqrt{2})$

As before, a non-relativistic potential can be obtained, producing the same orbits as equation (11):

$$V(r) = -\tfrac{1}{2}(vr^b/h^2)^2. \tag{12}$$

However, the relativistic and non-relativistic velocities of motion again differ.

3.3. *Non-planar motions*

When h and l are non-zero, we obtain the following exact solution of equations (6)–(8), for $b = 1$:

$$z = \sin\beta \cosh^{-1} R, \qquad R = r/m, \tag{13}$$

$$\phi = \cos\beta \cosh^{-1} R, \tag{14}$$

$$vt = \tfrac{1}{2}m^2[R(R^2-1)^{1/2} + |\cosh^{-1} R|], \tag{15}$$

$$s = t(1-v^2)^{1/2}, \tag{16}$$

where $m = (h^2 + l^2)^{1/2}$ is the minimal distance from the axis of symmetry, and $\beta = \tan^{-1} h/l$ is the angle of incidence on the plane $z = 0$. For $t < 0$ the particle is

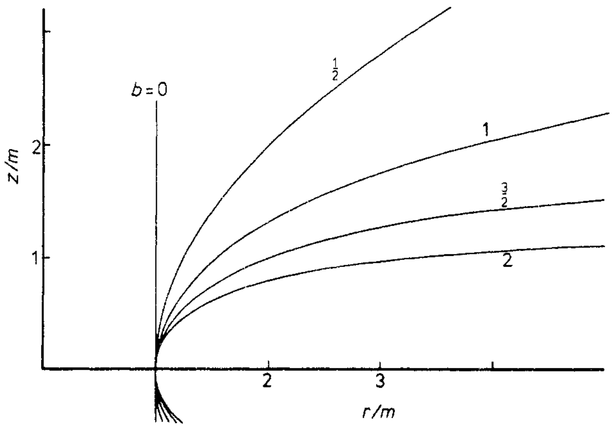

Figure 3. Orbits in planes containing the z axis for several values of b and for arbitrary m. Particles are repelled from the z axis.

approaching the z axis in a helical motion, and reaches the minimal radial distance $r = m$ when $t = 0$. For $t > 0$ a helical motion is found with increasing radius.

The solutions belonging to other values of b present similar basic features, but the corresponding mathematical expressions are rather involved. It can be shown that the non-planar orbits do not derive from any non-relativistic potential which is static and cylindrically symmetric.

4. Discussion

The source of gravitation of the system is concentrated around the axis of symmetry, as is seen from the scalar curvature, the square of the Ricci tensor, and the Kretschmann scalar:

$$R = 2b/r^{2(b+1)}, \qquad R^{\mu\nu}R_{\mu\nu} = R^2, \qquad R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 3R^2. \tag{17}$$

Since $b \geqslant 0$, all these quantities tend to zero at radial infinity. The same happens to the energy–momentum tensor, which is diagonal with components

$$T^0_0 = -T^1_1 = T^2_2 = T^3_3 = R/(16\pi) \geqslant 0. \tag{18}$$

We found in § 3 that a particle, once at rest, remains at rest. This is a consequence of the staticity of the metric with $g_{00} = 1$. Such a type of metric does not seem possible when electromagnetic fields are present.

The anisotropic state of stresses (18) is responsible for the peculiar, velocity-dependent gravitation originating from the scalar field. A radial acceleration field is found from equation (5), acting attractively upon the radial component of velocity of test particles, and repulsively upon the longitudinal component. We next compare the radial acceleration associated with the azimuthal velocity, $\ddot{r}_1 = r(r/a)^{1-2b}\dot{\phi}^2$, with its analogue in the case of rectilinear motion, $\ddot{r}_2 = r(r/a)\dot{\phi}^2$: since $\ddot{r}_2 \gtreqless \ddot{r}_1$ implies respectively $r \gtreqless a$, we find that the azimuthal component of the velocity is acted upon attractively when $r > a$, and repulsively when $r < a$. This explains the shapes of orbits in figure 2, drawn for $a = 1$.

One finds, from equation (6), that v represents the modulus of velocity of the test particle. All the results obtained for time-like geodesics are then also valid for light-like geodesics, provided one sets $v=1$.

A final comment concerns the physical interpretation of the metric (1). This metric is now obtained explicitly, following the prescriptions given in Teixeira *et al* (1976). We start from the line element (3), where λ is the linear density of matter in the weak field approximation, and obtain the intermediate solution

$$ds^2 = r^{4\mu}\, dt^2 - r^{-4\mu}[(dr^2 + dz^2)r^{8\lambda^2} + r^2\, d\phi^2], \tag{19}$$

$$R_{\alpha\beta} = -2\partial_\alpha S\, \partial_\beta S, \qquad S = 2c\lambda \ln r, \qquad \mu \equiv \lambda(1-c^2)^{1/2}, \tag{20}$$

where $c=$ constant. For weak fields, this intermediate solution corresponds to a linear density of matter μ, together with a linear source of scalar field $c\lambda$. The original vacuum solution (3) corresponds to the special value $c=0$, when the source of scalar field vanishes and $\mu=\lambda$. If we now start from the vacuum solution, fix λ and let c^2 increase, then the source of scalar field $|c\lambda|$ gradually increases, while the matter parameter μ gradually decreases. We interpret this process as gradual substitution of the original matter by an attractive scalar source. The substitution is completed when $c^2=1$, in which case $\mu=0$ and the line element becomes (1), with $b=4\lambda^2$. This fact strongly suggests that we should interpret (1) as the gravitation from the scalar field alone, at least in the weak field approximation.

References

Banerjee A and Dutta Choudhury S B 1977 *Phys. Rev.* D **10** 3062–4
Bekenstein J D 1974 *Ann. Phys., NY* **82** 535–47
—— 1975 *Ann. Phys., NY* **91** 75–82
Bronnikov K A 1978 *Gen. Rel. Grav.* **9** 271–5
Buchdahl H A 1978 *Gen. Rel. Grav.* **9** 59–70
Chung K C, Kodama T and Teixeira A F F 1977 *Phys. Rev.* D **16** 2412–6
Duncan C 1977 *Phys. Rev.* D **16** 1688–90
Dwight H B 1961 *Tables of Integrals and other Mathematical Data* 4th edn (New York: Macmillan)
Janis A I, Robinson D C and Winicour J 1969 *Phys. Rev.* **186** 1729–31
Kodama T, Chung K C and Teixeira A F F 1978 *Nuovo Cim.* B **46** 206–15
Teixeira A F F, Wolk I and Som M M 1974 *J. Math. Phys.* **15** 1756–9
—— 1975 *Phys. Rev.* D **12** 319–22
—— 1976 *J. Phys. A: Math. Gen.* **9** 53–8
Weyl H 1917 *Ann. Phys., Lpz* **54** 117–45